October 2007 S M Houghton

MATH3501 Modelling with Fluids

Example sheet 2

1. Find the equation for the path of a particle, released from $(1,1)$ at $t=0$, in the velocity field $\mathbf{u} = \left((1+t)^{-2}, (1+t)^{-1}, 0\right)$. Find the acceleration of the particle in the Eulerian and Lagrangian descriptions. How are these related?

2. A two dimensional flow is given by the velocity field
$$\mathbf{u} = \left(U, U\sin\left(x - Ut\right)\right),$$
where U is a constant.
   (a) Show that a particle released at the point (x_0, y_0) at time $t = t_0$ travels in a straight line at constant speed.
   (b) Find the equation for its path.
   (c) Write down the formula for the acceleration of a fluid particle, hence calculate the fluid acceleration at general point (x, y) at time t. Is your answer consistent with part (i)?
   (d) Show that the flow is incompressible.
   (e) Calculate the streamfunction, ψ, for the flow.

3. A two-dimensional flow is represented by the streamfunction $\psi(x, y)$ with $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$. Show that
   (a) the streamlines are given by $\psi = $ constant,
   (b) $|\mathbf{u}| = |\nabla\psi|$, so that the flow is faster where the streamlines are closer,
   (c) the volume flux crossing any curve between $\mathbf{x}_0$ and $\mathbf{x}_1$ is given by $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$, (Hint: the normal to the curve is given by $\mathbf{n} = \left(\frac{\mathrm{d}y}{\mathrm{d}s}, -\frac{\mathrm{d}x}{\mathrm{d}s}\right)$)
   (d) $\psi = $ constant on any *fixed* (i.e. stationary) boundary.

4. Verify that the two-dimensional flow given in Cartesian coordinates by
$$u = \frac{y-b}{(x-a)^2 + (y-b)^2}, \qquad v = \frac{a-x}{(x-a)^2 + (y-b)^2}$$
is incompressible and find the streamfunction $\psi(x, y)$. Sketch the streamlines and describe the flow.

5. Verify that the two-dimensional flow given in polar coordinates by
$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$
satisfies $\nabla\cdot\mathbf{u} = 0$, and find the streamfunction $\psi(r, \theta)$. Sketch the streamlines and describe the flow.
(Hint: in polar coordinates $\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$ and $u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, u_\theta = -\frac{\partial\psi}{\partial r}$.)

6. For each of the following two-dimensional flows (given in plane polar coordinates by $\mathbf{u} = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta$), show that $\nabla\cdot\mathbf{u} = 0$, calculate the streamfunction and sketch the streamlines.

   (a) $u_r = M/r$ and $u_\theta = 0$

   (b) $u_r = \frac{Ua^2}{r^2}\cos\theta$ and $u_\theta = \frac{Ua^2}{r^2}\sin\theta$

   (c) $u_r = 0$ and $u_\theta = f(r)$ where $f(r) = \Omega r$ for $r < a$ and $f(r) = \Omega a^2/r$ for $r > a$.

7. Verify that the axisymmetric flow given in cylindrical polar coordiantes by $u_r = -\frac{1}{2}\alpha r$, $u_z = \alpha z$ satisfies $\nabla\cdot\mathbf{u} = 0$, and find the Stokes streamfunction $\Psi(r, z)$ for the flow. Sketch the streamlines.
(Hint: $\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}$.)

8. (a) If $\mathbf{u} = \boldsymbol{\Omega}\times\mathbf{x}$, that is uniform rotation with angular velocity $\boldsymbol{\Omega}$, show that $\omega = 2\boldsymbol{\Omega}$.

   (b) For a two-dimensional flow $(u(x, y), v(x, y))$ with streamfunction $\psi(x, y)$ show that $\omega = \left(0, 0, -\nabla^2\psi\right)$.

Please send any comments, or corrections, to S M Houghton.

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