October 2007 S M Houghton

MATH3501 Modelling with Fluids

Example sheet 3

1. Calculate the vorticity of the velocity field

$$u = -\alpha x - yr\mathrm{e}^{3\alpha t}$$
$$v = -\alpha y + xr\mathrm{e}^{3\alpha t}$$
$$w = 2\alpha z$$

where $r^2 = x^2 + y^2$.
Calculate the velocity components u_r and u_θ in cylindrical polar coordinates. At time $t = 0$ we mark the fluid on the circle $r = a_0$ and $z = 0$. Show that at time t this fluid forms a circle with radius $r = a(t) = a_0\mathrm{e}^{-\alpha t}$.

2. Show that lines on which the streamfunction ψ are constant are perpendicular to lines of constant ϕ, where ϕ is the velocity potential.

3. A two-dimensional irrotational flow occupies the half-space $y < 0$ and is given by the velocity potential $\phi = \mathrm{e}^{ky}\sin kx$ with $k > 0$. Show that the flow is incompressible. Calculate the velocity field $\mathbf{u}$ and the streamfunction $\psi(x, y)$. Sketch the streamlines.

4. A cylinder of radius $r = a$ is held fixed in a steady stream of fluid flowing at velocity U. Write down the boundary condition that must be satisfied on the surface of the cylinder. Assuming that the flow remains irrotational, and that the velocity potential is of the form $\phi = f(r)\cos\theta$ (in cylindrical polars), find the velocity potential.
[Hint: $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$.]

5. Write down the velocity potential $\phi(r, \theta, z)$ for the axisymmetric flow produced by a point source of strength m located at the origin, in the presence of a uniform stream $(0, 0, U)$. Show that there is a stagnation point at $(0, 0, -a)$, where $a = (m/U)^{1/2}$. Show that the Stokes streamfunction is given by

$$\Psi = \frac{Ur^2}{2} - \frac{mz}{(z^2 + r^2)^{1/2}}.$$

Sketch the streamlines. From the sketch and the streamfunction show that ϕ represents the flow around a semi-infinite body whose radius tends to $2a$ far downstream.

6. Write down the velocity potential for a source of strength m located at position (a, b). Using the method of images find the flow field due to this source when walls are placed along the lines $x = 0$ and $y = 0$ (i.e. the source is near a corner).

7. A line vortex at position $(a, 0)$ has velocity potential $\phi = \kappa\tan^{-1}\left(\frac{y}{x-a}\right)$. Find the components of velocity and sketch the flow assuming $\kappa > 0$. A wall is placed along the y-axis

$(x = 0)$. Using the method of images, or otherwise, find the velocity potential for the flow.

When there are a collection of line vortices, they each move under the velocity field generated by the other vortices. Using this information describe the motion of the line vortex.

[Hint: $\frac{\mathrm{d}}{\mathrm{d}z} \tan^{-1} z = \frac{1}{1+z^2}$.]

8. A flow is given by the velocity potential

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \kappa\theta$$

Find the components of the velocity field in cylindrical coordinates. Locate the stagnation points in the flow. [You will need to consider the cases $\kappa > 2Ua$ and $\kappa < 2Ua$ separately.] Sketch and describe the fluid flow.

Please send any comments, or corrections, to S M Houghton.

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