November 2007 S M Houghton

MATH3501 Modelling with Fluids

Example sheet 4

1. How high can water rise up one's arm hanging in a river from a lazy gondola (1 ms^{-1})?

2. Waste water flows into a tank at 10^{-4}m^{3}s^{-1} and out of a short exit pipe of cross-section 4×10^{-5}m^2 into the air. In steady state, estimate how high above the pipe is the water in the tank?

3. A flat-bottomed barge closely fits a canal, so that while it travels very slowly it still generates a fast 5 m s^{-1} current under it. Estimate how much lower in the water is the barge as a result of this current?

   [Hint: Archimedes when stationary. Flow reduces pressure, so have to go deeper for same pressure on long bottom.]

4. Starting from the Euler momentum equation, show that $p/\rho + \frac{1}{2}|\mathbf{u}|^2 + gz$ is constant on streamlines (where z is measured upwards).

   A vertical pipe of length L and cross-sectional area a is connected to the base of a large tank with cross-sectional area A, where $A >> a$. Find the velocity of the fluid at the end of the pipe when the depth of fluid in the tank is h. Hence, show that the height of the fluid surface decreases as

   $$\frac{\partial h}{\partial t} \approx -\frac{a}{A}\sqrt{2g\,(h+L)}$$

   where terms of size a^2/A^2 are neglected. Find the time it takes to empty the tank (but not the pipe).

   Calculate the pressure high way down the pipe when the depth of fluid in the tank is h. Explain why the pressure at this point is lower than atmospheric pressure even though it is below the fluid surface.

5. Starting from the Euler momentum equation for a fluid of constant density with a body force given by potential force $-\nabla\chi$, show that for a fixed volume V enclosed by surface S

   $$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \frac{1}{2}\rho|\mathbf{u}|^2 \mathrm{d}V + \int_S H\mathbf{u}\cdot\mathbf{n}\mathrm{d}S = 0$$

   where $H = \frac{1}{2}\rho|\mathbf{u}|^2 + p + \chi$ is the Bernoulli quantity. Explain why H is sometimes called the transportable energy.

6. A water clock is an axisymmetric vessel with a small exit pipe in the bottom. Find the shape for which the water level falls equal heights in equal intervals of time. State any assumptions that you make.

7. A U-tube consists of two long uniform vertical tubes of cross-sectional area A_1 and A_2. These tubes are connected at the bas by a short tube of large cross-section. The system contains invisicd, incompressible fluid whose surface, at equilibrium, is a height h above the base.

   Derive the equation governing the nonlinear oscillations of the displacement ζ of the fluid surface in tube 2

   $$(h + r\zeta)\,\frac{\mathrm{d}^2\zeta}{\mathrm{d}t^2} + \frac{r}{2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)^2 + g\zeta = 0$$

   where $r = 1 - A_2/A_1$.

Please send any comments, or corrections, to S M Houghton.

smh@maths.leeds.ac.uk