November 2007 S M Houghton

MATH3501 Modelling with Fluids

Example sheet 5

1. Consider flow out of a vertical narrowing tube, where the exit is not much smaller than the cross-section. Let $A(z)$ be the cross-sectional area at height z with $A(0) = a$ and $A(z) \to A_0$ as $z \to \infty$.

   Explain why the volume flux $Q(t) = u(z,t)A(z)$ is independent of height. Hence, assuming that the flow is potential and purely in the z-direction show that the velocity potential is

   $$\phi = Q(t) \int_0^z \frac{\mathrm{d}\zeta}{A(\zeta)}.$$

   Apply (unsteady) Bernoulli to find the second order differential equation for $h(t)$.

2. *Sluice Gate*

   Define the Froude number F.

   By applying conservation of mass and Bernoulli on the free surface show that the upstream and downstream Froude numbers are related by

   $$\frac{\left(F_1^2+2\right)^3}{F_1^2} = \frac{\left(F_2^2+2\right)^3}{F_2^2}.$$

   Show that two solutions are possible. Find the value of the upstream Froude number that will allow for smooth transition between the two flow regimes.

3. An idealized river flows in a channel of rectangular cross-section, with a flat, horizontal bottom, and with width $w(x)$ varying slowly along the channel. Far upstream, the fluid velocity u has the constant value U, the depth of the water has the constant value H, and the width of the channel has the constant value W. Taking $u(x)$ to be constant over each cross-section of the channel, show [from mass conservation and Bernoulli] that

   $$\frac{W}{w} = \frac{u}{U}\left(1 + \frac{1}{2}F^2 - \frac{1}{2}F^2\frac{u^2}{U^2}\right)$$

   where $F^2 = U^2/gH$. Sketch this relationship. Observation of the river shows that $u(x)$ is steady and slowly varying and that far downstream $w \to W$ but $u \to V \neq U$. What can be deduced about W/w in the region of varying width? Find V and the depth far downstream.

4. Water from a large deep reservoir flows over a weir. The water is of depth d where the free surface has fallen to a level h below that far upstream in the reservoir. Assume that the depth of the water varies sufficiently slowly so that the velocity can be taken to be horizontal and uniform in depth. Show that the volume flux (per unit length normal to the flow) is $Q = d\sqrt{2gh}$. Check that this result is dimensionally consistent.

   From the condition that Q does not vary along the flow, and the condition that $h + d$ is a minimum at the crest of the weir [differentiate], show that $h = d/2$ at the crest.

   Deduce that $Q^2 = 8gL^3/27$ where L is the minimum value of $h + d$.

5. The velocity potential for flow past a cylinder, of radius a, with circulation $2\pi\kappa$ is given by

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \kappa\theta.$$

   Find the pressure distribution on the surface of the cylinder.

   Show that the force on the cylinder can be decomposed into a *drag force* F_D and a *lift force* F_L where

$$F_D = -\int_0^{2\pi} p\cos\theta a\mathrm{d}\theta \quad F_L = -\int_0^{2\pi} p\sin\theta a\mathrm{d}\theta.$$

   Hence, show that the drag force is always zero, but that there is a lift force proportional to the circulation around the cylinder.

6. Find the complex potential for flow past a cylinder of radius a with circulation $2\pi\kappa$.

   Locate the stagnation points in terms of the complex variable $z = x + \mathrm{i}y$.

   **Blasius theorem* The force on an object in a 2D flow with complex potential $w(z)$ is given by

$$F_x - \mathrm{i}F_y = \mathrm{i}\frac{\rho}{2}\int_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z.$$

   Calculate the force on the cylinder and compare your answer with the previous question.*

Please send any comments, or corrections, to S M Houghton.
smh@maths.leeds.ac.uk